Models for Spatially Dependent Areal Data

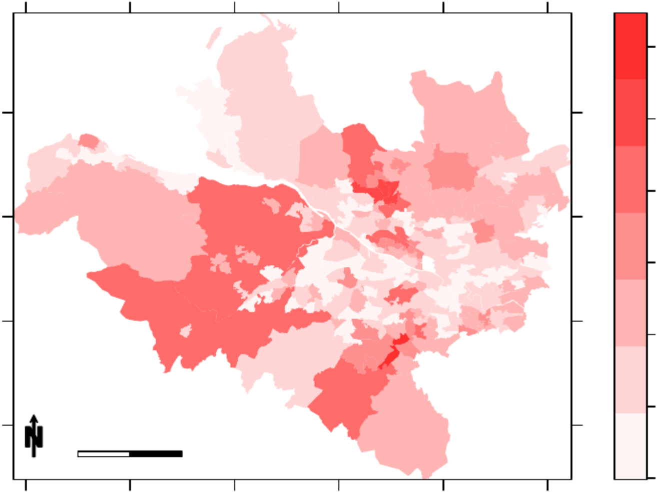
Models for Socio-Environmental Data

Chris Che-Castaldo, Mary B. Collins, N. Thompson Hobbs

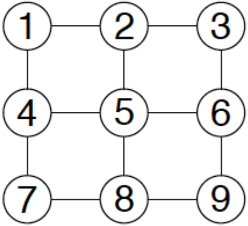
August 1, 2019

Most ecological data are spatial

Areal spatial processes



Areal data and proximity



$$\mathbf{W} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Possibilities include, but are not limited to:

- $w_{ij} = 1$ if i, j share a common boundary (possibly a common vertex)
- $w_{ij} = 1$ for m nearest neighbors.

Measures of regularity, clustering

Moran's I: similar to covariogram (see `spdep` package).

$$\hat{\mathbf{u}} = \mathbf{y} - \mathbf{X}\boldsymbol{\beta} \tag{1}$$

$$\mathbf{u} = \rho \mathbf{W}\mathbf{u} + \boldsymbol{\varepsilon},\ \boldsymbol{\varepsilon} \text{ multivariate normal}(\mathbf{0}, \sigma_\varepsilon^2 \mathbf{I}) \tag{2}$$

Moran's I: (3)

$$I = \frac{\hat{u}'}{||\hat{\mathbf{u}}||^2} \tag{4}$$

- $E(I) = -\frac{1}{n-1}$
- $I > E(I)$ implies clustering.
- $I < E(I)$ implies regularity.

Modeling areal data

Two general types of *spatial autoregressive models:*

- Simultaneous autoregressive models (SAR): less common in Bayesian analysis. Some references at end of lecture.
- Conditional autoregressive models (CAR): The probability of values estimated at any given location are conditional on neighboring values.

Conditional autoregressive model

$$
\begin{aligned}
\mathbf{y} &= g(\boldsymbol{\theta}, \mathbf{X}) + \boldsymbol{\eta} + \boldsymbol{\varepsilon} \\
\boldsymbol{\eta} &\sim \text{multivariate normal}(\mathbf{0}, \boldsymbol{\Sigma}) \\
\boldsymbol{\Sigma} &= \overbrace{\underbrace{\sigma^2(\mathbf{I} - \rho\mathbf{W})}_{\text{precision matrix}}{}^{-1}\mathbf{I}}^{\text{covariance matrix}} \\
\boldsymbol{\varepsilon} &\sim \text{multivariate normal}(\mathbf{0}, \sigma_{\varepsilon}^2\mathbf{I})
\end{aligned}
$$

1. ρ is an autocorrelation parameter.
2. Proximity matrix $\mathbf{W}$ must be symmetric.

Alternative notation

$$
\begin{aligned}
\mathbf{y} &\sim \text{multivariate normal}(g(\boldsymbol{\theta}, \mathbf{X}) + \boldsymbol{\eta}, \sigma_\varepsilon^2 \mathbf{I}) \\
\boldsymbol{\eta} &\sim \text{multivariate normal}(0, \boldsymbol{\Sigma})
\end{aligned}
$$

Conditional autoregressive model with row standardization

$$
\begin{aligned}
\mathbf{y} &\sim \text{multivariate normal}(g(\boldsymbol{\theta}, \mathbf{X}) + \boldsymbol{\eta}, \sigma_\varepsilon^2 \mathbf{I}) \\
\boldsymbol{\eta} &\sim \text{multivariate normal}(0, \boldsymbol{\Sigma}) \\
\boldsymbol{\Sigma} &= (\text{diag}(\mathbf{W}\mathbf{1}) - \rho \mathbf{W})^{-1}
\end{aligned}
$$

- Row standardization assures that $|\rho| < 1$
- $\mathbf{1}$ is a column vector of $1's$.
- $\mathbf{W1}$ is the sums of the rows
- diag$(\mathbf{W1})$ is a matrix with the sums of the rows on the diagonal and zeros elsewhere.
- Equivalent to dividing each element of $\mathbf{W}$ by the sum of the rows to obtain $\mathbf{W}_+$ and using $\boldsymbol{\Sigma} = \boldsymbol{\sigma}^{\mathbf{2}}(\mathbf{I} - \rho \mathbf{W}_+)^{-\mathbf{1}}$
- $\rho \sim \text{Beta}(18, 2)$ to favor values close to 1
- $\sigma^2 \sim \text{IG}(r, q)$.

CAR for non-negative observations

Let $\mathbf{\Sigma} = \sigma^2 \text{diag}(\mathbf{W1} - \rho\mathbf{W})^{-1}$

$$\begin{aligned} g(\boldsymbol{\beta}, \mathbf{X}) &= \text{exp}(\mathbf{X}\boldsymbol{\beta}) & (5) \\ \log(y) &\sim \text{multivariate normal}(\log(g(\boldsymbol{\beta}, \mathbf{X})) + \boldsymbol{\eta}, \boldsymbol{\sigma}_\varepsilon^2\mathbf{I}) & (6) \end{aligned}$$

CAR for counts

Let $\mathbf{\Sigma} = \sigma^2 \text{diag}(\mathbf{W1} - \rho \mathbf{W})^{-1}$

$$
\begin{aligned}
g(\boldsymbol{\beta}, \mathbf{X}) &= \text{exp}(\mathbf{X}\boldsymbol{\beta}) && (7)\\
y_i &\sim \text{Poisson}(\lambda_i) && (8)\\
\log(\boldsymbol{\lambda}) &\sim \text{multivariate normal}(\log(g(\boldsymbol{\beta}, \mathbf{X})) + \boldsymbol{\eta}, \boldsymbol{\sigma}_{\varepsilon}^2 \mathbf{I}) && (9)
\end{aligned}
$$

CAR for binary observations

Let $\Sigma = \sigma^2 \text{diag}(\mathbf{W1} - \rho \mathbf{W})^{-1}$

$$
\begin{aligned}
g(\boldsymbol{\beta}, \mathbf{X}) &= \text{logit}^{-1}(\mathbf{X}\boldsymbol{\beta}) && (10)\\
y_i &\sim \text{Bernoulli}(p_i) && (11)\\
\text{logit}(\mathbf{p}) &\sim \text{multivariate normal}(\text{logit}(g(\boldsymbol{\beta}, \mathbf{X})) + \boldsymbol{\eta}, \sigma_\varepsilon^2 \mathbf{I}) && (12)
\end{aligned}
$$

Take home

- Data taken over time or space are likely to be structured by physical and biological processes.
- Our deterministic model may account for this structure. However, if the *residuals* show correlation over time and/ or space, then we are obliged to model their covariance to assure that iid assumptions are met.
- Doing so requires estimating only a few more parameters, in most cases one or two, relative to the aspatial model.
- Fitting spatial models is computationally challenging.
- Deciding whether to use a spatial or aspatial model should probably be treated as a problem in model selection.

Further study

- Ver Hoef, J. M., E. E. Peterson, M. B. Hooten, E. M. Hanks, and M. J. Fortin. 2018. Spatial autoregressive models for statistical inference from ecological data. Ecological Monographs 88:36-59.
- Peterson, E.E., E.M. Hanks, M.B. Hooten, J.M. Ver Hoef, and M.-J. Fortin. (2019). Spatially structured statistical network models for landscape genetics. Ecological Monographs, 89: e01355.
- Finley, A. O., S. Banerjee, and B. P. Carlin. 2007. spBayes: An R package for univariate and multivariate hierarchical point-referenced spatial models. Journal of Statistical Software 19.
- Finley, A. O., S. Bsnerjee, and A. E. Gelfand. 2015. spBayes for Large Univariate and Multivariate Point-Referenced Spatio-Temporal Data Models. Journal of Statistical Software 63:1-28.
- Lee, D. 2013. CARBayes: An R Package for Bayesian Spatial Modeling with Conditional Autoregressive Priors. Journal of Statistical Software 55:1-24.
- Cressie, N., and C. K. Wikle. 2011. Statistics for spatio-temporal data. Wiley.