

Markov Chain Monte Carlo Using Gibbs Sampling

August 1, 2019

Accept-reject sampling methods like Metropolis and Metropolis-Hastings require lots of computation time because of the several steps required to calculate likelihoods, compute ratios and choose proposals vs current values. These multiple steps are tedious to code, and require "tuning" to achieve optimum efficiency. Gibbs sampling allows us to streamline that process by making proposals that are so smart that we retain *all* of them. This makes Gibbs updates much easier to code and faster to execute than accept-reject sampling methods. We can usefully illustrate Gibbs updates by showing how we would use them to estimate the posterior distribution of the mean of a normally distributed random variable. We will call this mean θ . Recall that draws of the random variable y_i from the normal distribution with mean θ arise as

$$y_i \sim \text{normal}(\theta, \zeta^2). \tag{1}$$

We can think of y_i , of course, as an observation on some socio-ecological process. For this example, we begin by assuming that the variance of the observations, ζ^2 is *known*. It is important to understand the "knowing" ζ^2 is not the same as calculating it as the variance of a sample dataset. Rather we are treating it here as a fully observed quantity, as if we had calculated it from *all* of the potential observations. In the following discussion, it is particularly important to keep in mind that ζ^2 is the variance of the distribution of the *observations* (y_i), not the variance of the distribution of the mean of the observations (θ).

We have prior information about θ ,

$$\theta \sim \text{normal}(\mu_0, \sigma_0^2). \tag{2}$$

This information might be informative or vague. Remember that μ_0 and σ_0^2 are numeric arguments. They are known. We have a data set \mathbf{y} with n observations. Given this information, we want to estimate the full-conditional distribution of θ . If we assume that the variance in the likelihood (ζ^2) is known then,

$$[\theta|\cdot] \propto \prod_{i=1}^n \text{normal}(y_i|\theta, \zeta^2) \text{normal}(\theta|\mu_0, \sigma_0^2). \quad (3)$$

We define μ_1 and σ_1^2 as the parameters of the full conditional posterior distribution of θ , that is

$$[\theta|\cdot] = \text{normal}(\mu_1, \sigma_1^2). \quad (4)$$

Note that σ_1^2 is the updated variance of the distribution of the *mean* not the variance of the distribution of the *observations*, which of course is ζ^2 . Note that we are treating ζ^2 as known. Equation 3 shows that we have a normal likelihood for the mean with known variance and a normal prior on the mean, which are conjugates. When this is the case, we can calculate the parameters of the full conditional posterior distribution of θ directly using the formulas

$$\mu_1 = \frac{\left(\frac{\mu_0}{\sigma_0^2} + \frac{\sum_{i=1}^n y_i}{\zeta^2}\right)}{\left(\frac{1}{\sigma_0^2} + \frac{n}{\zeta^2}\right)} \quad (5)$$

$$\sigma_1^2 = \left(\frac{1}{\sigma_0^2} + \frac{n}{\zeta^2}\right)^{-1}. \quad (6)$$

Notice that every quantity on the right hand side of these equations is known; μ_0 and σ_0^2 are known as priors. The y_i are observations in hand, and we are assuming (for now) that ζ^2 is known. So, we have all we need to know to make a draw from the distribution of θ using equation 4 because we know μ_1 and σ_1^2 . So, why wouldn't we just use equations 5 and 6 to estimate the parameters of the posterior of θ and be done with it? Because we must assume that ζ^2 is known, which is virtually never the case. Somehow we must learn about ζ^2 to estimate θ .

So, what about ζ^2 ? Again the observations arise from $y_i \sim \text{normal}(\theta, \zeta^2)$ and we seek to understand the full conditional¹ posterior distribution of ζ^2 . If we assume that θ is known, then

¹The distribution is conditional because we must know θ .

$$[\zeta^2|\cdot] \propto \prod_{i=1}^n \text{normal}(y_i|\theta, \zeta^2) \text{ inverse gamma}(\zeta^2|\alpha_0, \beta_0). \quad (7)$$

We define the parameters of the full conditional distribution of ζ^2 as α_1 and β_1 so that

$$[\zeta^2|\cdot] = \text{inverse gamma}(\alpha_1, \beta_1). \quad (8)$$

where α_0 and β_0 are the parameters of the prior distribution of ζ^2 . We have a normal likelihood with a known mean and unknown variance and an inverse gamma prior on the variance. When this is true we can calculate the parameters of the full-conditional distribution of ζ^2 using

$$\alpha_1 = \alpha_0 + \frac{n}{2} \quad (9)$$

$$\beta_1 = \beta_0 + \frac{\sum_{i=1}^n (y_i - \theta)^2}{2}. \quad (10)$$

Again, remember that β_0 and α_0 are known arguments to priors, so in practice they would be numeric. It follows that all quantities on the right hand side of equations 9 and 10 are known. It is also critical to understand that β_1 is a *scale* parameter.

It might seem that we have tied ourselves in a knot. We need to know ζ^2 to estimate θ and we need to know θ to estimate ζ^2 . This is just the kind of problem that MCMC can solve because at each step in the chain we pretend all of the parameters save one are *known*. Equations 5 - 10 give us all we need to construct a very fast sampler for θ and ζ^2 . Define k as the iteration in the chain. So, element 100 in the chain is indexed by $k = 100$. Be sure you understand that k is a superscript not an exponent. The algorithm is:

1. Use the current value of $\zeta^{2(k)}$ to calculate $\mu_1^{(k+1)}$ and $\sigma_1^{2(k+1)}$ from equations 5 and 6. Make a draw from $\theta^{(k+1)} \sim \text{normal}(\mu_1^{(k+1)}, \sigma_1^{2(k+1)})$ and store it in the chain.
2. Use the updated value of $\theta^{(k+1)}$ to calculate $\alpha_1^{(k+1)}$ and $\beta_1^{(k+1)}$ using equations 9 and 10. Make a draw from $\zeta^{2(k+1)} \sim \text{inverse gamma}(\alpha_1^{(k+1)}, \beta_1^{(k+1)})$ and store it in the chain.
3. Repeat 1-2.

A sufficient number of repetitions usually converges on the posterior distributions of θ and ζ^2 much more quickly than if we used an accept-reject sampling method like Metropolis-Hastings or Metropolis. However, the estimates would be the same.