Derivation of beta-binomial conjugate relationship

We seek to the posterior distribution of the parameter ϕ, the probability of success on n trials with y successes:

$$[\phi|y] \propto \underbrace{\begin{pmatrix} y \\ n \end{pmatrix} \phi^{y}(1-\phi)^{n-y}}_{\text{binomial likelihood}} \underbrace{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\phi^{\alpha-1}(1-\phi)^{\beta-1}}_{\text{beta prior}}. \qquad (1)$$

- Drop the normalizing constants:

$$[\phi|y] \propto \underbrace{\phi^{y}(1-\phi)^{n-y}}_{\text{binomial likelihood}} \underbrace{\phi^{\alpha-1}(1-\phi)^{\beta-1}}_{\text{beta prior}} \qquad (2)$$

- Simplify:

$$[\phi|y] \quad \propto \quad \phi^{y+\alpha-1}(1-\phi)^{\beta+n-y-1} \qquad (3)$$

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- Recognizing from the beta prior in 2

$$[\phi|y] \propto \phi^{\overbrace{y+\alpha}^{\text{the new }\alpha}-1}(1-\phi)^{\overbrace{\beta+n-y}^{\text{the new }\beta}-1}, \tag{4}$$

we let $\alpha_{new} = y + \alpha, \quad \beta_{new} = \beta + n - y$ and substitute into the normalizing constant, obtaining $\frac{\Gamma(\alpha_{new}+\beta_{new})}{\Gamma(\alpha_{new})\Gamma(\beta_{new})}$.

- Multiply eq. 4 by the new normalizing constant $\frac{\Gamma(\alpha_{new}+\beta_{new})}{\Gamma(\alpha_{new})\Gamma(\beta_{new})}$
- Voila, a new beta distribution informed by the prior and the data:

$$[\phi|y] = \frac{\Gamma(\alpha_{new}+\beta_{new})}{\Gamma(\alpha_{new})\Gamma(\beta_{new})}\phi^{\alpha_{new}-1}(1-\phi)^{\beta_{new}-1} \tag{5}$$

$$= \text{beta}(y+\alpha, \beta+n-y) \tag{6}$$

Derivation of beta-binomial conjugate prior

Also see `https://www.youtube.com/watch?v=hKYvZF9wXkk`

where $B(\alpha, \beta) = \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right)^{-1}$